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INFINITE SERIES RELATION FROM A MODULAR TRANSFORMATION FORMULA FOR THE GENERALIZED EISENSTEIN SERIES

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Abstract. In 1970s, B. C. Berndt proved a transformation formula for a large class of functions that includes the classical Dedekind eta function. From this formula, he evaluated several classes of infinite series and found a lot of interesting infinite series identities. In this paper, using his formula, we find new infinite series identities.

1. Introduction and preliminaries

In 1970s, B. C. Berndt [2, 3] found a lot of infinite series identities using a modular transformation formula for the generalized Eisenstein series. Some of his results have been stated in the Notebooks of Ramanujan [7] or are generalizations of formulas of Ramanujan. Recently he suggested that one could find more new infinite series identities using his modular transformation formula in [3]. In fact, continuing his work, the author derived a lot of new series relation between infinite series [4, 5, 6]. In this paper, we obtain more infinite series identities, some of which are compared with series relations in [2, 3].

The basic notations are as follows. For a complex w, we choose the branch of the argument for a complex w defined by $-\pi \leq \arg w < \pi$. Let $e(w) = e^{2\pi i w}$ and $V\tau = V(\tau) = \frac{a\tau+b}{c\tau+d}$ always denote a modular transformation with $c > 0$ for every complex τ. Let $r = (r_1, r_2)$ and $h = (h_1, h_2)$ denote real vectors, and the associated vectors R and H are defined by

$$R = (R_1, R_2) = (ar_1 + cr_2, br_1 + dr_2)$$

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and

$$H = (H_1, H_2) = (dh_1 - bh_2, -ch_1 + ah_2).$$

Let λ denote the characteristic function of the integers. For a real number x, $[x]$ denotes the greatest integer less than or equal to x and $\{x\} := x - [x]$. For real x, y and $\mathrm{Re}(s) > 1$, let

$$\psi(x, y, s) := \sum_{n+y>0} \frac{e(nx)}{(n+y)^s}.$$

If x is an integer and y is not an integer, then $\psi(x, y, s) = \zeta(s, \{y\})$, where $\zeta(s, x)$ is the Hurwitz zeta-function. The function $\psi(x, y, s)$ can be analytically continued to the entire s-plane except for a possible simple pole at $s = 1$ when x is an integer. Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, the upper half-plane. For $\tau \in \mathbb{H}$ and an arbitrary complex numbers s, define

$$A(\tau, s; r, h) := \sum_{m+r_1>0} \sum_{n-h_2>0} \frac{e\left(mh_1 + ((m + r_1)\tau + r_2)(n - h_2)\right)}{(n - h_2)^{1-s}}.$$

Let

$$H(\tau, s; r, h) := A(\tau, s; r, h) + e\left(s/2\right) A(\tau, s; -r, -h).$$

We now state the principal theorem for our results.

Theorem 1.1. [2]. *Let $Q = \{\tau \in \mathbb{C} \mid \mathrm{Re}(\tau) > -d/c\}$ and $\varrho = c\{R_2\} -d\{R_1\}$. Then for $\tau \in Q$ and all s,*

$$\begin{aligned}
&(c\tau + d)^{-s} H(V\tau, s; r, h) = H(\tau, s; R, H)\\
&-\lambda(r_1)e(-r_1h_1)(c\tau + d)^{-s}\Gamma(s)(-2\pi i)^{-s}\left(\psi(h_2, r_2, s) + e\left(s/2\right)\psi(-h_2, -r_2, s)\right)\\
&+\lambda(R_1)e(-R_1H_1)\Gamma(s)(-2\pi i)^{-s}\left(\psi(H_2, R_2, s) + e\left(-s/2\right)\psi(-H_2, -R_2, s)\right)\\
&+(2\pi i)^{-s} L(\tau, s; R, H),
\end{aligned}$$

where

$$\begin{aligned}
&L(\tau, s; R, H)\\
&:= \sum_{j=1}^{c'} e(-H_1(j + [R_1] - c) - H_2([R_2] + 1 + [(jd + \varrho)/c] - d))\\
&\qquad \cdot \int_C u^{s-1} \frac{e^{-(c\tau+d)(j-\{R_1\})u/c}}{e^{-(c\tau+d)u} - e(cH_1 + dH_2)} \frac{e^{\{(jd+\varrho)/c\}u}}{e^u - e(-H_2)} du,
\end{aligned}$$

where C is a loop beginning at $+\infty$, proceeding in the upper half-plane, encircling the origin in the positive direction so that $u = 0$ is the only zero of

$$\left(e^{-(c\tau+d)u} - e(cH_1 + dH_2)\right)\left(e^u - e(-H_2)\right)$$

lying "inside" the loop, and then returning to $+\infty$ in the lower half plane. Here, we choose the branch of u^s with $0 < \arg u < 2\pi$.

REMARK 1.1. Theorem 1.1 is true for $\tau \in Q$. But, after the evaluation of $L(\tau, s; R, H)$ for an integer s, it will be valid for all $\tau \in \mathbb{H}$ by analytic continuation.

We shall use the Bernoulli polynomials $B_n(x)$, $n \geq 0$, defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \ (|t| < 2\pi).$$

The n-th Bernoulli number B_n, $n \geq 0$, is defined by $B_n = B_n(0)$. Put $\bar{B}_n(x) = B_n(\{x\}),\ n \geq 0$. Recall that $B_{2n+1} = 0,\ n \geq 1$, and that $B_{2n+1}(1/2) = 0,\ n \geq 0$. The following formulas [1] are helpful ;

$$B_n(1 - x) = (-1)^n B_n(x),$$

$$B_n\left(\frac{1}{2}\right) = -(1 - 2^{1-n}) B_n,\ n \geq 0.$$

We also use the Euler polynomials $E_n(x)$, $n \geq 0$, defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \ (|t| < \pi).$$

The Euler numbers E_n are defined by

$$E_n := 2^n E_n\left(\frac{1}{2}\right),\ n \geq 0.$$

Put $\bar{E}_n(x) = E_n(\{x\}),\ n \geq 0$. Recall also that $E_{2n+1}(1/2) = 0,\ n \geq 0$.

2. Infinite series identities

From now on, we let V a modular transformation corresponding to

$$\begin{pmatrix} 1 & -1 \\ c & 1-c \end{pmatrix}$$

for $c > 0$. Put $r = (r_1, r_2/c)$. Then

$$R_1 = r_1 + r_2,\ R_2 = -r_1 - r_2 + \frac{r_2}{c}.$$

Replacing $c\tau + 1 - c$ by z, we have

$$V\tau = \frac{1}{c} - \frac{1}{cz},\ \tau = 1 - \frac{1}{c} + \frac{1}{c} z.$$

If $\tau \in Q$, then Re $z > 0$ and $z \in \mathbb{H}$. By Remark 1.1, we shall put $z = \pi i/\alpha$ for a positive real number α. In this section, we consider three cases of $h = (h_1, h_2)$, i.e., $h = (1/2, 1/2),\ (1/2, 0)$ and $(0, 1/2)$. We also suppose that r_1 and r_2 are integers. In this case, $\lambda(r_1) = \lambda(R_1) = 1$. By Theorem 1.1, we have, for any integer m and $z \in \mathbb{H}$ with Re $z > 0$,

$$\begin{aligned} z^m H(V\tau, -m; r, h) &= H(\tau, -m; R, H) + (2\pi i)^m L(\tau, -m; R, H) \\ &\quad + \lim_{s \to -m} (-2\pi i)^{-s} \Gamma(s) \left(-\Phi_+(s, r, h) + \Phi_-(s, R, H) \right), \end{aligned} \tag{2.1}$$

where

$$\Phi_+(s, r, h) := e(-r_1 h_1) z^{-s} \left(\psi\left(h_2, \frac{r_2}{c}, s\right) + e\left(\frac{s}{2}\right) \psi\left(-h_2, -\frac{r_2}{c}, s\right) \right)$$

and

$$\Phi_-(s, R, H) := e(-R_1 H_1) \left(\psi(H_2, R_2, s) + e\left(-\frac{s}{2}\right) \psi(-H_2, -R_2, s) \right).$$

We need the following equations to compute equation (2.1). For r_1 and r_2 integers,

$$\begin{aligned} H(V\tau, s; r, h) &= e(-r_1 h_1) \sum_{n - h_2 > 0} \frac{e(h_1 + (V\tau + r_2/c)(n - h_2))}{(n - h_2)^{1-s}(1 - e(h_1 + V\tau(n - h_2)))} \\ &\quad + e^{\pi i s} e(-r_1 h_1) \sum_{n + h_2 > 0} \frac{e(-h_1 + (V\tau - r_2/c)(n + h_2))}{(n + h_2)^{1-s}(1 - e(-h_1 + V\tau(n + h_2)))} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} H(\tau, s; R, H) &= e(-R_1 H_1) \sum_{n - H_2 > 0} \frac{e(H_1 + (\tau + R_2)(n - H_2))}{(n - H_2)^{1-s}(1 - e(H_1 + \tau(n - H_2)))} \\ &\quad + e^{\pi i s} e(-R_1 H_1) \sum_{n + H_2 > 0} \frac{e(-H_1 + (\tau - R_2)(n + H_2))}{(n + H_2)^{1-s}(1 - e(-H_1 + \tau(n + H_2)))}. \end{aligned} \tag{2.3}$$

It is easy to see that, for $x \notin \mathbb{Z}$,

$$\begin{aligned} &\psi(1/2, x, s) = (-1)^{[x]}(2^{1-s}\zeta(s, \{x\}/2) - \zeta(s, \{x\})), \\ &\psi(-1/2, -x, s) = (-1)^{[x]+1}(2^{1-s}\zeta(s, (1 - \{x\})/2) - \zeta(s, 1 - \{x\})). \end{aligned} \tag{2.4}$$

If x is an integer, then

$$\psi(\pm 1/2, \pm x, s) = (-1)^x (2^{1-s} - 1)\zeta(s). \tag{2.5}$$

For Re $s < 0$ and $0 < x \leq 1$([8], p. 37),

$$\Gamma(s)\zeta(s, x) = \frac{(2\pi)^s}{\sin(\pi s)} \sum_{n=1}^{\infty} \frac{\sin(2\pi n x + \pi s/2)}{n^{1-s}}. \tag{2.6}$$

Let $\Psi_0(x)$ be the digamma function defined by

$$\Psi_0(x) = \frac{d}{dx}\Gamma(x).$$

For brevity, we let

$$\mathcal{Z}_\pm(s, x) := \zeta(s, x) \pm \zeta(s, 1 - x) \tag{2.7}$$

and let

$$\mathfrak{z}_{\pm}(s,x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s} \pm \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1-x)^s} \tag{2.8}$$

for $0 < x < 1$ and $\mathrm{Re}\, s > 0$. Then $\mathfrak{z}_{\pm}(s,x)$ can be analytically continued to an entire function.

THEOREM 2.1. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integers k, r_2 and for any positive even integer c,*

$$\begin{aligned}
&\alpha^{-k} \sum_{n=0}^{\infty} \frac{2\cos(\pi r_2(2n+1)/c)}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}+1)} \\
&= (-1)^{r_2}(-\beta)^{-k} \sum_{n=0}^{\infty} \frac{2\cos(\pi r_2(2n+1)/c)}{(2n+1)^{2k+1}(e^{(\beta+\pi i)(2n+1)/c}+1)} \\
&+ \frac{(-1)^{r_2}}{4} \sum_{j=1}^{c} (-1)^{j+[(j+r_2)/c]} \sum_{\ell=0}^{2k} \frac{E_\ell(j/c)\bar{E}_{2k-\ell}((j+r_2)/c)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell} + I_1(k),
\end{aligned}$$

where if r_2/c is not an integer, then

$$I_1(k) := \begin{cases} \frac{(-1)^{[r_2/c]}}{2}\Gamma(-2k)((-\beta)^k - (-1)^{r_2}\alpha^k)\mathfrak{z}_-(-2k, \{\frac{r_2}{c}\}) & \text{if } k < 0, \\ \frac{(-1)^{[r_2/c]+1}}{2}\left(((-1)^{r_2}-1)\log\cot\left(\frac{\pi}{2}\left\{\frac{r_2}{c}\right\}\right) + ((-1)^{r_2}+1)\frac{\pi i}{2}\right) & \text{if } k = 0, \\ \alpha^{-k}\sum_{n=0}^{\infty}\frac{e^{-\pi i r_2(2n+1)/c}}{(2n+1)^{2k+1}} + (-1)^{r_2+1}(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{e^{\pi i r_2(2n+1)/c}}{(2n+1)^{2k+1}} & \text{if } k > 0, \end{cases}$$

and if r_2/c is an integer, then

$$I_1(k) := \begin{cases} (-1)^{r_2/c+1}(1-2^{-2k-1})((-\beta)^{-k} - \alpha^{-k})\zeta(2k+1) & \text{if } k \neq 0, \\ \frac{(-1)^{r_2/c}}{4}\left(\log\frac{\beta}{\alpha} - \pi i\right) & \text{if } k = 0. \end{cases}$$

Proof. Let $h = (1/2, 1/2)$ and $m = 2k$ in (2.1). We have from (2.2) that

$$\begin{aligned}
&H(V\tau, -2k; r, h) \\
&= (-1)^{r_1} \sum_{n=0}^{\infty} \frac{e(r_2(2n+1)/(2c))}{2^{-2k-1}(2n+1)^{2k+1}} \cdot \frac{-e((1-1/z)(2n+1)/(2c))}{1+e((1-1/z)(2n+1)/(2c))} \\
&\quad + (-1)^{r_1} \sum_{n=0}^{\infty} \frac{e(-r_2(2n+1)/(2c))}{2^{-2k-1}(2n+1)^{2k+1}} \cdot \frac{-e((1-1/z)(2n+1)/(2c))}{1+e((1-1/z)(2n+1)/(2c))}
\end{aligned}$$

$$= (-1)^{r_1+1}2^{2k+2} \sum_{n=0}^{\infty} \frac{\cos(\pi r_2(2n+1)/c)}{(2n+1)^{2k+1}(1+e^{-\pi i(1-1/z)(2n+1)/c})}. \tag{2.9}$$

Since c is even, $H_1 \equiv 0 \pmod 1$ and $H_2 \equiv 1/2 \pmod 1$. Thus it follows from (2.3) that

$$H(\tau, -2k; R, H) = -2^{2k+1} \sum_{n=0}^{\infty} \frac{e^{\pi i R_2(2n+1)} + e^{-\pi i R_2(2n+1)}}{(2n+1)^{2k+1}(e^{\pi i(1-z)(2n+1)/c}+1)}$$

$$= (-1)^{r_1+r_2+1}2^{2k+2} \sum_{n=0}^{\infty} \frac{\cos(\pi r_2(2n+1)/c)}{(2n+1)^{2k+1}(e^{\pi i(1-z)(2n+1)/c}+1)}. \tag{2.10}$$

We see that

$$\frac{e^{-zuj/c}}{e^{-zu}+1} = \frac{1}{2}\sum_{n=0}^{\infty} E_n\left(\frac{j}{c}\right)\frac{(-zu)^n}{n!},$$
$$\frac{e^{\{(j(1-c)+\varrho)/c\}u}}{e^u+1} = \frac{1}{2}\sum_{n=0}^{\infty} \bar{E}_n\left(\frac{j+\varrho}{c}\right)\frac{u^n}{n!},$$

and

$$\left[\frac{j(1-c)+\varrho}{c}\right] = -j - \left[\frac{r_2}{c}\right] + \left[\frac{j+[R_2]}{c}\right].$$

Then, by the residue theorem,

$$\begin{aligned} L(\tau,-2k;R,H) &= \frac{1}{4}\sum_{j=1}^{c} e\left(-\frac{1}{2}\left([R_2]+c+\left[\frac{j(1-c)+\varrho}{c}\right]\right)\right) \\ &\quad\cdot \int_C u^{-2k-1}\sum_{n=0}^{\infty} E_n\left(\frac{j}{c}\right)\frac{(-zu)^n}{n!}\cdot\sum_{m=0}^{\infty}\bar{E}_m\left(\frac{j+\varrho}{c}\right)\frac{u^m}{m!}du \\ &= \frac{(-1)^{r_1+r_2}}{2}\pi i\sum_{j=1}^{c}(-1)^{j+[(j+r_2)/c]} \\ &\quad\cdot\sum_{\ell=0}^{2k}\frac{E_\ell(j/c)}{\ell!}\cdot\frac{\bar{E}_{2k-\ell}((j+r_2)/c)}{(2k-\ell)!}(-z)^\ell. \end{aligned} \tag{2.11}$$

For Re $s<0$ and $x\notin\mathbb{Z}$, apply (2.4) and (2.6) to obtain that

$$\begin{aligned} &\Gamma(s)\left(\psi\left(\frac{1}{2},x,s\right)+e\left(\pm\frac{s}{2}\right)\psi\left(-\frac{1}{2},-x,s\right)\right) \\ &= 2(\pm\pi)^s e^{\pi is/2}\sum_{n=0}^{\infty}\frac{e^{\mp\pi ix(2n+1)}}{(2n+1)^{1-s}}. \end{aligned} \tag{2.12}$$

In case of $s=0$, using the expansions at $s=0$,

$$\begin{aligned} 2^{1-s} &= 2-2\log 2s+\cdots, \\ \zeta(s,x) &= \frac{1}{2}-x+\left(\log\Gamma(x)-\frac{1}{2}\log 2\pi\right)s+\cdots, \\ e^{\pi is} &= 1+\pi is+\cdots, \end{aligned}$$

we have, for $x\notin\mathbb{Z}$,

$$\begin{aligned} &\lim_{s\to 0}\Gamma(s)\left(\psi\left(\frac{1}{2},x,s\right)+e\left(\pm\frac{s}{2}\right)\psi\left(-\frac{1}{2},-x,s\right)\right) \\ &= (-1)^{[x]}\left(\log\cot\left(\frac{\pi}{2}\{x\}\right)\mp\frac{1}{2}\pi i\right). \end{aligned} \tag{2.13}$$

Employing the expansion at $s=0$,

$$\Gamma(s) = \frac{1}{s}+\gamma+\cdots$$

and using (2.5), we obtain that

$$\lim_{s\to 0}\Gamma(s)(1+e^{-\pi is}-z^{-s}(1+e^{\pi is}))=2\log z-2\pi i. \tag{2.14}$$

Now put (2.9) – (2.14) in (2.1) and let $z=\pi i/\alpha$. Then we prove the theorem. □

If $c=2$ in Theorem 2.1, then, equating the real part and the imaginary part, respectively, we obtain Theorem 4.7 in [3] and Proposition 4.5 in [2].

THEOREM 2.2. *Let $\alpha,\ \beta>0$ with $\alpha\beta=\pi^2$. Then, for any integer k, r_2 and for any positive odd integer c,*

$$\begin{aligned}
&\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos\left((2n+1)\pi r_2/c\right)}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}+1)}\\
&=(-1)^{r_2}2^{-2k-1}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{2\cos\left(2\pi nr_2/c\right)}{n^{2k+1}(e^{(\beta+\pi i)2n/c}+1)}\\
&\quad+\frac{(-1)^{r_2}}{2}\sum_{j=1}^{c}(-1)^{j+1}\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j}{c}\right)\bar{B}_{2k+1-\ell}\left(\frac{j+r_2}{c}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell}+I_2(k),
\end{aligned}$$

where if r_2/c is not an integer, then

$$I_2(k):=\begin{cases}\frac{(-1)^{[r_2/c]}}{2}(-\beta)^k\Gamma(-2k)\mathfrak{Z}_-(-2k,\{\frac{r_2}{c}\}) & \\ \quad+\frac{(-1)^{r_2+1}}{2}\alpha^k\Gamma(-2k)\mathcal{Z}_+(-2k,\{\frac{r_2}{c}\}) & if\ k<0,\\ \frac{(-1)^{[r_2/c]}}{2}\left(\log\cot\left(\frac{\pi}{2}\left\{\frac{r_2}{c}\right\}\right)-\frac{1}{2}\pi i\right)+\frac{(-1)^{r_2}}{2}\log(1-e^{2\pi ir_2/c}) & if\ k=0,\\ \alpha^{-k}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi ir_2/c}}{(2n+1)^{2k+1}}+(-1)^{r_2+1}2^{-2k-1}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{e^{2\pi inr_2/c}}{n^{2k+1}} & if\ k>0,\end{cases}$$

and if r_2/c is an integer, then

$$I_2(k):=\begin{cases}(-1)^{r_2+1}2^{-2k-1}((-\beta)^{-k}-(2^{2k+1}-1)\alpha^{-k})\zeta(2k+1) & if\ k\neq 0,\\ \frac{(-1)^{r_2}}{4}\left(\log\frac{\beta}{\alpha}+4\log 2-\pi i\right) & if\ k=0.\end{cases}$$

Proof. Let $h=(1/2,1/2)$ and $m=2k$ in (2.1). Since c is odd, $H_1\equiv 1/2 \pmod 1$ and $H_2\equiv 0 \pmod 1$. By the similar way as we derived equation (2.10) and (2.11), we obtain that

$$H(\tau,-2k;R,H)=(-1)^{r_1+r_2+1}2\sum_{n=0}^{\infty}\frac{\cos(2\pi r_2n/c)}{n^{2k+1}(e^{2\pi i(1-z)n/c}+1)} \tag{2.15}$$

and

$$\begin{aligned}
L(\tau,-2k;R,H)&=\frac{1}{2}\sum_{j=1}^{c}e\left(-\frac{1}{2}(j+[R_1]-c)\right)\\
&\quad\cdot\int_C u^{-2k-2}\sum_{n=0}^{\infty}E_n\left(\frac{j}{c}\right)\frac{(-zu)^n}{n!}\cdot\sum_{m=0}^{\infty}\bar{B}_m\left(\frac{j+r_2}{c}\right)\frac{u^m}{m!}du
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{r_1+r_2+1}\pi i \sum_{j=1}^{c}(-1)^j \\
&\quad \cdot \sum_{\ell=0}^{2k+1} \frac{E_\ell(j/c)}{\ell!} \cdot \frac{\bar{B}_{2k+1-\ell}((j+r_2)/c)}{(2k+1-\ell)!}(-z)^\ell. \qquad (2.16)
\end{aligned}
$$

It is easy to see that, for $0 < x < 1$,

$$
\lim_{s\to 0} \Gamma(s)(\zeta(s,\{x\}) + e^{-\pi i s}\zeta(s, 1-\{x\})) = -\log(1-e^{2\pi i\{x\}}) \qquad (2.17)
$$

and

$$
\begin{aligned}
&\lim_{s\to 0} \Gamma(s)(1+e^{-\pi i s} - z^{-s}(2^{1-s}-1)(1+e^{\pi i s})) \\
&= 2\log z + 4\log 2 - 2\pi i. \qquad (2.18)
\end{aligned}
$$

Apply (2.4)–(2.6), (2.15)–(2.18) to (2.1) and let $z = \pi i/\alpha$ to complete the proof. □

If $c = 1$ and $k \neq 0$ in Theorem 2.2, then Theorem 5.6 in [3] follows.

Theorem 2.3. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer k, r_2 and for any positive even integer c,*

$$
\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty} \frac{2\sin((2n+1)\pi r_2/c)}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}+1)} \\
&= (-1)^{r_2}(-\beta)^{-k-1/2}\sum_{n=0}^{\infty} \frac{2\sin((2n+1)\pi r_2/c)}{(2n+1)^{2k+2}(e^{(\beta+\pi i)(2n+1)/c}+1)} \\
&\quad + \frac{(-1)^{r_2+1}}{4}\pi\sum_{j=1}^{c}(-1)^{j+[\frac{j+r_2}{c}]}\sum_{\ell=0}^{2k+1} \frac{E_\ell\left(\frac{j}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j+r_2}{c}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^\ell \alpha^{k-\ell+1/2} \\
&\quad + I_3(k),
\end{aligned}
$$

where if r_2/c is not an integer, then

$$
I_3(k) := \begin{cases}
(-1)^{[r_2/c]+k+1}\frac{1}{2}\Gamma(-2k-1)(\beta^{k+1/2}+(-1)^{r_2}(-\alpha)^{k+1/2}) & \\
\quad \cdot \mathfrak{Z}_+(-2k-1, \{\frac{r_2}{c}\}) & if\ k < -1, \\
\frac{(-1)^{[r_2/c]}}{2}\left(\beta^{-1/2}+(-1)^{r_2}(-\alpha)^{-1/2}\right)\Big(\Psi_0(\{\frac{r_2}{c}\}) + \Psi_0(1-\{\frac{r_2}{c}\}) & \\
\qquad -\Psi_0(\frac{1}{2}\{\frac{r_2}{c}\}) - \Psi_0(\frac{1}{2}-\frac{1}{2}\{\frac{r_2}{c}\}) - 2\log 2\Big) & if\ k = -1, \\
\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{ie^{-(2n+1)\pi i r_2/c}}{(2n+1)^{2k+2}} & \\
\qquad +(-1)^{r_2}(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{ie^{(2n+1)\pi i r_2/c}}{(2n+1)^{2k+2}} & if\ k \geq 0,
\end{cases}
$$

and if r_2/c is an integer, then

$$
I_3(k) := \begin{cases}
0 & if\ k < 0, \\
\frac{(-1)^{r_2/c+1}\pi}{2(2k+1)!}(2^{2k+2}-1)((-\beta)^{k+1/2}+\alpha^{k+1/2})\zeta(-2k-1) & if\ k \geq 0.
\end{cases}
$$

Proof. Let $z = \pi i/\alpha$, $h = (1/2, 1/2)$ and $m = 2k+1$ in (2.1). All details of the proof are similar to those in the proof of Theorem 2.1 except for $m = -1$. For $0 < x < 1$, $\zeta(s,x)$ has the expansion at $s = 1$,

$$\zeta(s,x) = \frac{1}{s-1} - \Psi_0(x) + \cdots . \tag{2.19}$$

Then, using (2.4) and (2.5), we obtain that for $x \notin \mathbb{Z}$,

$$\begin{aligned}
&\lim_{s\to 1}\left(\psi\left(\frac{1}{2}, x, s\right) + e\left(\pm\frac{s}{2}\right)\psi\left(-\frac{1}{2}, -x, s\right)\right)\\
&= (-1)^{[x]}\Bigg(\Psi_0(\{x\}) - \Psi_0(\frac{1}{2}\{x\})\\
&\qquad +\Psi_0(1-\{x\}) - \Psi_0(\frac{1}{2}(1-\{x\})) - 2\log 2\Bigg)
\end{aligned}$$

and for $x \in \mathbb{Z}$,

$$\lim_{s\to 1}\left(\psi\left(\frac{1}{2}, x, s\right) + e\left(\pm\frac{s}{2}\right)\psi\left(-\frac{1}{2}, -x, s\right)\right) = 0.$$

□

Corollary 2.4. *For any integer k and for any positive even integer c,*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}+1)}\\
&= (-1)^{c/2}(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+2}(e^{(\beta+\pi i)(2n+1)/c}+1)}\\
&\quad -\frac{(-1)^{c/2}}{8}\pi\sum_{j=1}^{c}(-1)^{j+[\frac{j}{c}+\frac{1}{2}]}\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j}{c}+\frac{1}{2}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+1/2} + \mathcal{I}_3(k),
\end{aligned}$$

where

$$\mathcal{I}_3(k) := \begin{cases}
\frac{(-1)^{k+1}}{2^{2k+2}}(\beta^{k+1/2} + (-1)^{c/2}(-\alpha)^{k+1/2})\Gamma(-2k-1) & \\
\qquad\qquad \cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{-2k-1}} & \text{if } k < -1,\\
\frac{1}{4}\left(\alpha^{1/2} - (-1)^{c/2}(-\beta)^{1/2}\right) & \text{if } k = -1,\\
\frac{1}{2}(\alpha^{-k-1/2} - (-1)^{c/2}(-\beta)^{-k-1/2})\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+2}} & \text{if } k \geq 0.
\end{cases}$$

Proof. Put $r_2/c = 1/2$ in Theorem 2.3. □

For $c = 2$, Corollary 2.4 yields Corollary 4.19 in [3].

THEOREM 2.5. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer k, r_2 and for any positive odd integer c,*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2\sin\left((2n+1)\pi r_2/c\right)}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}+1)}\\
&=(-1)^{r_2}2^{-2k-2}(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{2\sin\left(2\pi n r_2/c\right)}{n^{2k+2}(e^{(\beta+\pi i)2n/c}+1)}\\
&\quad+\frac{(-1)^{r_2}}{2}\pi\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{E_\ell\left(\frac{j}{c}\right)\bar{B}_{2k+2-\ell}\left(\frac{j+r_2}{c}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+1/2}+I_4(k),
\end{aligned}$$

where if r_2/c is not an integer, then

$$I_4(k):=\begin{cases}
(-1)^{[r_2/c]+k+1}\frac{1}{2}\beta^{k+1/2}\Gamma(-2k-1)\mathfrak{Z}_+(-2k-1,\{\frac{r_2}{c}\})\\
+(-1)^{r_2+k+1}\frac{1}{2}(-\alpha)^{k+1/2}\mathcal{Z}_-(-2k-1,\{\frac{r_2}{c}\}) & \textit{if } k<-1,\\
\frac{(-1)^{[r_2/c]}}{2}\beta^{-1/2}\left(\Psi_0(\{\frac{r_2}{c}\})+\Psi_0(1-\{\frac{r_2}{c}\})\right.\\
\qquad\left.-\Psi_0(\frac{1}{2}\{\frac{r_2}{c}\})-\Psi_0(\frac{1}{2}-\frac{1}{2}\{\frac{r_2}{c}\})-2\log 2\right)\\
+\frac{(-1)^{r_2}}{2}(-\alpha)^{-1/2}(\pi\cot(\pi\{\frac{r_2}{c}\})+\pi i) & \textit{if } k=-1,\\
\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{ie^{-(2n+1)\pi i r_2/c}}{(2n+1)^{2k+2}}\\
\qquad+(-1)^{r_2}2^{-2k-2}\beta^{-k-1/2}\sum_{n=1}^{\infty}\frac{ie^{2\pi i r_2 n/c}}{n^{2k+2}} & \textit{if } k\ge 0,
\end{cases}$$

and if r_2/c is an integer, then

$$I_4(k):=\begin{cases}
0 & \textit{if } k<-1,\\
\frac{(-1)^{r_2}}{2}\beta^{1/2} & \textit{if } k=-1,\\
\frac{(-1)^{r_2/c}\pi}{2(2k+1)!}((1-2^{2k+2})(-\beta)^{k+1/2}+\alpha^{k+1/2})\zeta(-2k-1) & \textit{if } k\ge 0.
\end{cases}$$

Proof. Let $z=\pi i/\alpha$, $h=(1/2,1/2)$ and $m=2k+1$ in (2.1). The proof is similar to the proof of Theorem 2.2 besides $m=-1$. Employing (2.19) and the formula

$$\Psi_0(1-x)-\Psi_0(x)=\pi\cot(\pi x),$$

it follows that for $x\notin\mathbb{Z}$,

$$\lim_{s\to 1}\left(\psi\left(0,x,s\right)+e\left(-\frac{s}{2}\right)\psi\left(0,-x,s\right)\right)=\pi\cot(\pi\{x\})+\pi i$$

and for $x\in\mathbb{Z}$,

$$\lim_{s\to 1}\left(\psi\left(0,x,s\right)+e\left(-\frac{s}{2}\right)\psi\left(0,-x,s\right)\right)=\pi i.$$

□

If r_2/c is not an integer, then Theorem 2.2 and Theorem 2.5 have the same flavor as Theorem 5.11 and Theorem 5.12, respectively, in [3].

We obtain more theorems from (2.1) by putting different values in $h = (h_1, h_2)$. The proofs of these theorems can be done by the similar methods using equations in the proofs of above theorems. Now we state our results without details of proof.

THEOREM 2.6. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer k, r_2 and for any positive even integer c,*

$$\begin{aligned}
&\alpha^{-k}\sum_{n=1}^{\infty}\frac{2\cos(2\pi n r_2/c)}{n^{2k+1}(e^{(\alpha-\pi i)2n/c}+1)}\\
&=(-1)^{r_2}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{2\cos(2\pi n r_2/c)}{n^{2k+1}(e^{(\beta+\pi i)2n/c}+1)}\\
&\quad-(-1)^{r_2}2^{2k+1}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{B_\ell\left(\frac{j}{c}\right)\bar{B}_{2k+2-\ell}\left(\frac{j+r_2}{c}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+1}+I_5(k),
\end{aligned}$$

where if r_2/c is not an integer, then

$$I_5(k) := \begin{cases} 2^{2k}((-\beta)^k-(-1)^{r_2}\alpha^k)\Gamma(-2k)\mathcal{Z}_+(-2k,\{\frac{r_2}{c}\}) & \text{if } k<0,\\ -\log\left(1-e^{-2\pi i r_2/c}\right)+(-1)^{r_2}\log\left(1-e^{2\pi i r_2/c}\right) & \text{if } k=0,\\ \alpha^{-k}\displaystyle\sum_{n=1}^{\infty}\frac{e^{-2\pi i n r_2/c}}{n^{2k+1}}+(-1)^{r_2+1}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{e^{2\pi i n r_2/c}}{n^{2k+1}} & \text{if } k>0,\end{cases}$$

and if r_2/c is an integer, then

$$I_5(k) := \begin{cases} (\alpha^{-k}-(-\beta)^{-k})\zeta(2k+1) & \text{if } k\neq 0,\\ \frac{1}{2}\log\frac{\beta}{\alpha}-\frac{1}{2}\pi i & \text{if } k=0.\end{cases}$$

Proof. Let $z = \pi i/\alpha$, $h = (1/2, 0)$ and $m = 2k$ in (2.1). □

For c odd, if we put $h = (1/2, 0)$, $m = 2k$ and $z = \pi i/\alpha$ in (2.1), then the complex conjugate of the identity in Theorem 2.2 follows.

COROLLARY 2.7. *For any integer k and for any positive even integer c,*

$$\begin{aligned}
&\alpha^{-k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k+1}(e^{(\alpha-\pi i)2n/c}+1)}\\
&=(-1)^{c/2}(-\beta)^{-k}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2k+1}(e^{(\beta+\pi i)2n/c}+1)}\\
&\quad-(-1)^{c/2}2^{2k}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+2}\frac{B_\ell\left(\frac{j}{c}\right)\bar{B}_{2k+2-\ell}\left(\frac{j}{c}+\frac{1}{2}\right)}{\ell!(2k+2-\ell)!}(-\pi i)^\ell\alpha^{k-\ell+1}+\mathcal{I}_5(k),
\end{aligned}$$

where

$$\mathcal{I}_5(k) := \begin{cases} (2^{-2k-1}-2^{-1})(\alpha^{-k}-(-1)^{r_2}(-\beta)^{-k})\zeta(2k+1) & \text{if } k\neq 0,\\ ((-1)^{r_2}-1)\log\sqrt{2} & \text{if } k=0.\end{cases}$$

Proof. Put $r_2/c = 1/2$ in Theorem 2.6. $\square$

THEOREM 2.8. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer k, r_2 and for any positive even integer c,*

$$\begin{aligned}
&\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{2i\sin(2\pi n r_2/c)}{n^{2k+2}(e^{(\alpha-\pi i)2n/c}+1)}\\
&=(-1)^{r_2}(-\beta)^{-k-1/2}\sum_{n=1}^{\infty}\frac{2i\sin(2\pi n r_2/c)}{n^{2k+2}(e^{(\beta+\pi i)2n/c}+1)}\\
&\quad+(-1)^{r_2}2^{2k+2}\sum_{j=1}^{c}(-1)^j\sum_{\ell=0}^{2k+3}\frac{B_\ell\left(\frac{j}{c}\right)\bar{B}_{2k+3-\ell}\left(\frac{j+r_2}{c}\right)}{\ell!(2k+3-\ell)!}(-\pi i)^{\ell}\alpha^{k-\ell+3/2}+I_6(k),
\end{aligned}$$

where if r_2/c is not an integer, then

$$I_6(k) := \begin{cases}
2^{2k+1}((-1)^{r_2}\alpha^{k+1/2}-(-\beta)^{k+1/2})\Gamma(-2k-1)\mathcal{Z}_-(-2k-1,\{\frac{r_2}{c}\}) & if\ k<-1,\\
\frac{1}{2}\alpha^{1/2}(i\cot(\pi\{\frac{r_2}{c}\})+1)+(-1)^{r_2+1}\frac{1}{2}(-\beta)^{1/2}(i\cot(\pi\{\frac{r_2}{c}\})-1) & if\ k=-1,\\
-\alpha^{-k-1/2}\sum_{n=1}^{\infty}\frac{e^{-2\pi i n r_2/c}}{n^{2k+2}}+(-1)^{r_2+1}(-\beta)^{-k-1/2}\sum_{n=1}^{\infty}\frac{e^{2\pi i n r_2/c}}{n^{2k+2}} & if\ k\ge 0,
\end{cases}$$

and if r_2/c is an integer, then

$$I_6(k) := \begin{cases}
0 & if\ k<-1,\\
\frac{1}{2}(\alpha^{1/2}+(-\beta)^{1/2}) & if\ k=-1,\\
-(\alpha^{-k-1/2}+(-\beta)^{-k-1/2})\zeta(2k+2) & if\ k\ge 0.
\end{cases}$$

Proof. Let $z = \pi i/\alpha$, $h = (1/2, 0)$ and $m = 2k+1$ in (2.1). $\square$

For c odd, if $h = (1/2, 0)$, $m = 2k+1$ and $z = \pi i/\alpha$ in (2.1), then we have the complex conjugate of the equation in Theorem 2.5. Theorem 2.6 and Theorem 2.8 should be compared with Theorem 3.20 and Theorem 3.21 in [5], respectively.

THEOREM 2.9. *Let $\alpha,\ \beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer k, r_2 and for any positive integer c,*

$$\begin{aligned}
&\alpha^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi r_2/c)}{(2n+1)^{2k+1}(e^{(\alpha-\pi i)(2n+1)/c}-1)}\\
&=(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{2\cos((2n+1)\pi r_2/c)}{(2n+1)^{2k+1}(e^{(\beta+\pi i)(2n+1)/c}-1)}\\
&\quad-\frac{1}{4}\sum_{j=1}^{c}(-1)^{[\frac{j+r_2}{c}]}\sum_{\ell=0}^{2k}\frac{E_\ell\left(\frac{j}{c}\right)\bar{E}_{2k-\ell}\left(\frac{j+r_2}{c}\right)}{\ell!(2k-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell}+I_7(k),
\end{aligned}$$

where if r_2/c is not an integer, then

$$I_7(k) := \begin{cases}
\frac{(-1)^{[r_2/c]}}{2}(\alpha^k-(-\beta)^k)\Gamma(-2k)\mathfrak{Z}_-(-2k,\{\frac{r_2}{c}\}) & if\ k<0,\\
\frac{(-1)^{[r_2/c]}}{2}\pi i & if\ k=0,\\
-\alpha^{-k}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i r_2/c}}{(2n+1)^{2k+1}}+(-\beta)^{-k}\sum_{n=0}^{\infty}\frac{e^{(2n+1)\pi i r_2/c}}{(2n+1)^{2k+1}} & if\ k>0,
\end{cases}$$

and if r_2/c is an integer, then

$$I_7(k) := \begin{cases} (-1)^{r_2/c}(2^{-2k-1}-1)(\alpha^{-k}-(-\beta)^{-k})\zeta(2k+1) & \text{if } k \neq 0, \\ \frac{(-1)^{r_2/c+1}}{4}\left(\log\frac{\beta}{\alpha}-\pi i\right) & \text{if } k = 0. \end{cases}$$

Proof. Let $z = \pi i/\alpha$, $h = (0, 1/2)$ and $m = 2k$ in (2.1). □

If $c = 1$, then Theorem 2.9 yields Theorem 4.7 in [3]. If $c = 2$ in Theorem 2.9, then we also obtain Theorem 4.7 in [3] and Proposition 4.5 in [2].

THEOREM 2.10. *Let α, $\beta > 0$ with $\alpha\beta = \pi^2$. Then, for any integer k, r_2 and for any positive integer c,*

$$\begin{aligned} &\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{2i\sin((2n+1)\pi r_2/c)}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}-1)} \\ &= (-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{2i\sin((2n+1)\pi r_2/c)}{(2n+1)^{2k+2}(e^{(\beta+\pi i)(2n+1)/c}-1)} \\ &\quad -\frac{1}{4}\sum_{j=1}^{c}(-1)^{[\frac{j+r_2}{c}]}\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j+r_2}{c}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell+1}\alpha^{k-\ell+1/2} + I_8(k), \end{aligned}$$

where if r_2/c is not an integer, then

$$I_8(k) := \begin{cases} \frac{(-1)^{[r_2/c]}}{2}((-\beta)^{k+1/2}-\alpha^{k+1/2})\Gamma(-2k-1)\mathfrak{z}_+(-2k-1,\{\frac{r_2}{c}\}) & \text{if } k < -1, \\ \frac{(-1)^{[r_2/c]}}{2}((-\beta)^{-1/2}-\alpha^{-1/2})\left(\Psi_0(\{\frac{r_2}{c}\})+\Psi_0(1-\{\frac{r_2}{c}\})\right. & \\ \qquad \left.-\Psi_0(\frac{1}{2}\{\frac{r_2}{c}\})-\Psi_0(\frac{1}{2}-\frac{1}{2}\{\frac{r_2}{c}\})-2\log 2\right) & \text{if } k = -1, \\ \alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)\pi i r_2/c}}{(2n+1)^{2k+2}}+(-\beta)^{-k-1/2}\sum_{n=0}^{\infty}\frac{e^{(2n+1)\pi i r_2/c}}{(2n+1)^{2k+2}} & \text{if } k \geq 0, \end{cases}$$

and if r_2/c is an integer, then

$$I_8(k) := \begin{cases} 0 & \text{if } k < 0, \\ (-1)^{r_2/c}(1-2^{-2k-2})(\alpha^{-k-1/2}+(-\beta)^{-k-1/2})\zeta(2k+2) & \text{if } k \geq 0. \end{cases}$$

Proof. Let $z = \pi i/\alpha$, $h = (0, 1/2)$ and $m = 2k+1$ in (2.1). □

In case of c even, Theorem 2.9 and Theorem 2.10 should be compared with Theorem 2.1 and Theorem 2.3, respectively.

COROLLARY 2.11. *For any integer k and for any positive integer c,*

$$\begin{aligned} &\alpha^{-k-1/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+2}(e^{(\alpha-\pi i)(2n+1)/c}-1)} \\ &= (-\beta)^{-k-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2k+2}(e^{(\beta+\pi i)(2n+1)/c}-1)} \\ &\quad +\frac{\pi}{8}\sum_{j=1}^{c}(-1)^{[\frac{j}{c}+\frac{1}{2}]}\sum_{\ell=0}^{2k+1}\frac{E_\ell\left(\frac{j}{c}\right)\bar{E}_{2k+1-\ell}\left(\frac{j}{c}+\frac{1}{2}\right)}{\ell!(2k+1-\ell)!}(-\pi i)^{\ell}\alpha^{k-\ell+1/2} + \mathcal{I}_8(k), \end{aligned}$$

where

$$\mathcal{I}_8(k) := \begin{cases} -2^{-2k-2} i((-\beta)^{k+1/2} - \alpha^{k+1/2})\Gamma(-2k-1)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{-2k-1}} & if\ k < -1, \\ \frac{1}{4}((-\beta)^{1/2} - \alpha^{1/2}) & if\ k = -1, \\ \frac{1}{2}\left((-\beta)^{-k-1/2} - \alpha^{-k-1/2}\right)\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+2}} & if\ k \geq 0. \end{cases}$$

Proof. Put $r_2/c = 1/2$ in Theorem 2.10. □

If $c = 1$ in Corollary 2.11, then Corollary 4.19 in [3] is obtained.

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